

# SOLUTIONS OF COMPLEX DIFFERENTIAL EQUATION HAVING ZEROS ON PRE-GIVEN SEQUENCES

JANNE GRÖHN

**ABSTRACT.** Behavior of solutions of  $f'' + Af = 0$  is discussed under the assumption that  $A$  is analytic in  $\mathbb{D}$  and  $\sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |A(z)| < \infty$ , where  $\mathbb{D}$  is the unit disc of the complex plane. As a main result it is shown that such differential equation may admit a non-trivial solution whose zero-sequence does not satisfy the Blaschke condition. This gives an answer to an open question in the literature.

It is also proved that  $\Lambda \subset \mathbb{D}$  is the zero-sequence of a non-trivial solution of  $f'' + Af = 0$  where  $|A(z)|^2 (1 - |z|^2)^3 dm(z)$  is a Carleson measure if and only if  $\Lambda$  is uniformly separated. As an application an old result, according to which there exists a non-normal function which is uniformly locally univalent, is improved.

## 1. INTRODUCTION

Let  $\mathcal{H}(\mathbb{D})$  be the collection of analytic functions in the unit disc  $\mathbb{D}$  of the complex plane  $\mathbb{C}$ . This research concerns zero-sequences of non-trivial solutions ( $f \neq 0$ ) of

$$f'' + Af = 0 \tag{1}$$

under the assumption  $A \in H_2^\infty$ , which means that  $A \in \mathcal{H}(\mathbb{D})$  and  $\|A\|_{H_2^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^2 |A(z)| < \infty$ . In particular, we are interested in the following question:

- (Q) Is it true that the zero-sequence  $\{z_n\}$  of any non-trivial solution of (1) satisfies the Blaschke condition  $\sum_n (1 - |z_n|) < \infty$  if  $A \in H_2^\infty$ ?

Question (Q) relates to so-called Blaschke-oscillatory equations, and appears in [12, pp. 61–62]. Note that the characterization [12, Lemma 3] of Blaschke-oscillatory equations does not provide an immediate answer to (Q). Moreover, it is known that all non-trivial solutions of (1) may lie outside the Nevanlinna class  $N$ , even if  $A \in H_2^\infty$  [12, pp. 57–58].

We will make repeated use of [15, Theorem 6.1], which connects non-trivial solutions of (1) to a locally univalent meromorphic function in  $\mathbb{D}$ . If  $\|A\|_{H_2^\infty} \leq 1$  then all non-trivial solutions of (1) vanish at most once in  $\mathbb{D}$  by [19, Theorem I], and if  $\|A\|_{H_2^\infty} > 1$  then non-trivial solutions may have infinitely many zeros [13]. The condition  $A \in H_2^\infty$  is equivalent to the fact that zero-sequences of non-trivial solutions of (1) are separated with respect

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to the pseudo-hyperbolic metric [25, Theorems 3–4], by a constant depending on  $\|A\|_{H_2^\infty}$ , and hence zero-sequences *almost* satisfy the Blaschke condition [3, p. 162]. Many sufficient coefficient conditions implying an affirmative answer to (Q) are known. For coefficient conditions placing all solutions of (1) to the Nevanlinna class, see [11, 21], and for coefficient conditions placing all solutions to some Hardy space, see [8, 10, 21, 24]. For a more direct approach to zero-sequences of solutions of (1), see [9].

## 2. RESULTS

As the main result, we prove that there is  $A \in H_2^\infty$  such that (1) admits a non-trivial solution whose zero-sequence does not satisfy the Blaschke condition. This answers (Q) in the negative. Actually, we show that a non-trivial solution may vanish on any pre-given sequence of sufficiently small density.

We also obtain a complete description of zero-sequences of solutions of (1) in the case that  $A$  satisfies a condition stronger than  $A \in H_2^\infty$ . In Section 2.2, we consider an application concerning normal meromorphic functions.

**2.1. Zero-sequences of solutions.** The sequence  $\Lambda = \{z_n\}$  of points in  $\mathbb{D}$  is said to be uniformly separated if

$$\inf_k \prod_{n \neq k} \left| \frac{z_n - z_k}{1 - \bar{z}_n z_k} \right| > 0,$$

while  $\Lambda$  is called separated if there exists a constant  $\delta = \delta(\Lambda) > 0$  such that  $\varrho(z_n, z_k) = |z_n - z_k|/|1 - \bar{z}_n z_k| > \delta$  for any  $n \neq k$ . Unless otherwise stated, separation is understood with respect to the pseudo-hyperbolic metric.

Let  $\Delta(\zeta, r) = \{z \in \mathbb{D} : \varrho(z, \zeta) < r\}$  be the (open) pseudo-hyperbolic disc of radius  $0 < r < 1$ , centered at  $\zeta \in \mathbb{D}$ , and let  $n(\Lambda, \zeta, r)$  be the counting function for those points in  $\Lambda$  which lie in  $\Delta(\zeta, r)$ . The lower and upper uniform densities of  $\Lambda$  are

$$\begin{aligned} D^-(\Lambda) &= \liminf_{r \rightarrow 1^-} \left( \log \frac{1}{1-r} \right)^{-1} \inf_{\zeta \in \mathbb{D}} \int_0^r n(\Lambda, \zeta, s) ds, \\ D^+(\Lambda) &= \limsup_{r \rightarrow 1^-} \left( \log \frac{1}{1-r} \right)^{-1} \sup_{\zeta \in \mathbb{D}} \int_0^r n(\Lambda, \zeta, s) ds, \end{aligned} \quad (2)$$

respectively. For a comprehensive treatment of these densities, and their connection to interpolation and sampling, see [3, Chapters 6–7] and [28, Chapter 3]. See also the seminal papers [26, 27] by Seip.

**Theorem 1.** *If  $\Lambda \subset \mathbb{D}$  is a separated sequence for which  $D^+(\Lambda) < 1$ , then there exists  $A = A(\Lambda) \in H_2^\infty$  such that (1) admits a non-trivial solution which vanishes at all points on  $\Lambda$ .*

*Conversely, if  $A \in H_2^\infty$  and  $f$  is a non-trivial solution of (1) whose zero-sequence is  $\Lambda \subset \mathbb{D}$ , then  $\Lambda$  is separated, and contains at most one point if  $\|A\|_{H_2^\infty} \leq 1$ , while otherwise*

$$D^+(\Lambda) \leq (2\pi + 1) \left( 1 - \frac{2\|A\|_{H_2^\infty}^{1/2}}{\|A\|_{H_2^\infty} + 1} \right)^{1/2} \left( 1 - \left( 1 - \frac{2\|A\|_{H_2^\infty}^{1/2}}{\|A\|_{H_2^\infty} + 1} \right)^{1/2} \right)^{-2}.$$

Let  $\lesssim$  denote a one-sided estimate up to a constant and write  $\simeq$  for a two-sided estimate up to constants. In the first part of Theorem 1 we construct a solution  $f$  of (1) which vanishes on  $\Lambda$  (but has other zeros also). The idea is to find an auxiliary function  $g \in \mathcal{H}(\mathbb{D})$  such that

$$(1 - |z|^2)^\beta |g(z)| \simeq \varrho(z, \Lambda^*), \quad z \in \mathbb{D}, \quad (3)$$

for some constant  $0 < \beta < \infty$ , and for some separated sequence  $\Lambda^* \supset \Lambda$  which satisfies  $D^+(\Lambda^*) < 1$ . Such function  $g$  exists in the literature. Then,  $f = ge^h$  is a solution of (1) for  $A \in H_2^\infty$  if  $h$  belongs to the Bloch space and solves a certain interpolation problem. Note that the additional assumption  $D^+(\Lambda) < 1$  in Theorem 1 holds for any sequence  $\Lambda \subset \mathbb{D}$  which is separated by a constant sufficiently close to one [3, Lemma 9, p. 190]. The second part of Theorem 1 follows immediately by combining [3, Lemma 9, p. 190], [19, Theorem I] and [25, Theorem 3]. It implies that  $D^+(\Lambda) \rightarrow 0^+$  at the linear rate as  $\|A\|_{H_2^\infty} \rightarrow 1^+$ .

The following construction is due to Seip [26, pp. 214–215]. For  $a > 1$  and  $b > 0$ , let  $\Lambda = \Lambda(a, b)$  be the image of  $\{a^j(bk + i)\}_{j,k \in \mathbb{Z}}$  under the Cayley transform (conformal map from the upper half-plane onto  $\mathbb{D}$ ). The set  $\Lambda \subset \mathbb{D}$  satisfies  $D^-(\Lambda) = D^+(\Lambda) = 2\pi/(b \log a)$  by [27, p. 23]. In particular, there exists  $g \in \mathcal{H}(\mathbb{D})$  which satisfies (3) for  $\Lambda^* = \Lambda$  and  $\beta = 2\pi/(b \log a)$ . Now  $\Lambda$  is a separated sequence which behaves (essentially) as badly as possible in terms of the Blaschke condition; recall that the lower uniform density of any separated Blaschke sequence is zero by (2). As a straightforward consequence of Theorem 1, we obtain:

**Corollary 2.** *Let  $\Lambda = \Lambda(a, b) \subset \mathbb{D}$  be as above, and let  $2\pi/(b \log a) < 1$ . Then, there exists  $A = A(a, b) \in H_2^\infty$  such that (1) admits a non-trivial solution whose zero-sequence is  $\Lambda$ , and hence does not satisfy the Blaschke condition.*

Let  $0 < p < \infty$ . A positive Borel measure  $\mu$  on  $\mathbb{D}$  is called a (bounded)  $p$ -Carleson measure provided that

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(Q(I))}{|I|^p} < \infty, \quad Q(I) = \{re^{i\theta} : 1 - |I| \leq r < 1, e^{i\theta} \in I\},$$

where the supremum is taken over all subarcs  $I \subset \partial\mathbb{D}$  and  $|I|$  denotes the length of  $I$  (normalized so that  $|\partial\mathbb{D}| = 1$ ). These measures can be described in conformally invariant terms [1, Lemma 2.1]. In fact, the positive Borel measure  $\mu$  is a  $p$ -Carleson measure if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^p d\mu(z) < \infty. \quad (4)$$

For  $p = 1$  the condition (4) characterizes the classical Carleson measures, which were invented to study interpolation by bounded analytic functions. See [5] for a general reference.

There are two types of measures which play a role in this study. First, let  $\delta_z$  be the Dirac mass at the point  $z \in \mathbb{D}$ . We consider separated sequences

$\Lambda \subset \mathbb{D}$  for which

$$\sum_{z_n \in \Lambda} (1 - |z_n|)^p \delta_{z_n} \quad (5)$$

is a  $p$ -Carleson measure. Such sequences are uniformly separated for any  $0 < p \leq 1$ . Second, we consider functions  $A \in \mathcal{H}(\mathbb{D})$  for which

$$d\mu_{A,p}(z) = |A(z)|^2 (1 - |z|^2)^{2+p} dm(z) \quad (6)$$

is a  $p$ -Carleson measure. Here  $dm(z)$  is the Lebesgue area measure on  $\mathbb{D}$ . We write  $\mu_A = \mu_{A,1}$  for short. Such functions satisfy  $A \in H_2^\infty$  by the subharmonicity of  $|A|^2$ . The effect of the parameter  $p$  is more evident when second primitives of  $A \in \mathcal{H}(\mathbb{D})$  are considered [23, Theorem 3.2]: (6) is a  $p$ -Carleson measure if and only if the second primitive of  $A$  belongs to  $Q_p$ . The space  $Q_p$  consists of those  $f \in \mathcal{H}(\mathbb{D})$  for which  $|f'(z)|^2 (1 - |z|^2)^p dm(z)$  is a  $p$ -Carleson measure. For  $1 < p < \infty$ ,  $Q_p$  is the Bloch space while  $Q_{p_1} \subsetneq Q_{p_2} \subsetneq Q_1 = \text{BMOA}$  for any  $0 < p_1 < p_2 < 1$ . See [1, 4, 20] and the references therein, for more details.

**Theorem 3.** *Let  $0 < p \leq 1$ . If  $\Lambda \subset \mathbb{D}$  is a separated sequence such that (5) is a  $p$ -Carleson measure, then there exists  $A = A(\Lambda) \in \mathcal{H}(\mathbb{D})$  such that (6) is a  $p$ -Carleson measure and (1) admits a non-trivial solution whose zero-sequence is  $\Lambda$ .*

Theorems 1 and 3 improve [7, Corollary 7], which states that any uniformly separated sequence can appear as the zero-sequence of a non-trivial solution of (1) where  $A \in H_2^\infty$ , i.e., the second primitive of  $A$  is in the Bloch space. Theorem 1 shows that we can prescribe zero-sequences of strictly positive uniform density under the same coefficient condition while Theorem 3 implies that any sufficiently separated sequence can be prescribed such that the second primitive of  $A$  belongs to  $Q_p$  for fixed  $0 < p \leq 1$ . When prescribing infinite zero-sequences to non-trivial solutions of (1), we cannot expect that the coefficient  $A$  is even close to be bounded. The breaking point lies inside  $H_2^\infty$ : if  $A \in \mathcal{H}(\mathbb{D})$  and there exists  $0 < R < 1$  such that  $(1 - |z|^2)^2 |A(z)| \leq 1$  for  $R < |z| < 1$ , then all non-trivial solutions of (1) have at most finitely many zeros in  $\mathbb{D}$  [25, Theorem 1].

By Theorem 3, we obtain a complete description of zero-sequences of non-trivial solutions of (1) in the case that  $d\mu_A(z) = |A(z)|^2 (1 - |z|^2)^3 dm(z)$  is a Carleson measure.

**Corollary 4.** *A sequence  $\Lambda \subset \mathbb{D}$  is the zero-sequence of a non-trivial solution of (1) where  $\mu_A$  is a Carleson measure if and only if  $\Lambda$  is uniformly separated.*

In Corollary 4, all zero-sequences are uniformly separated by [9, Corollary 3]. The converse assertion follows from Theorem 3 by taking  $p = 1$ . The following observation concerns the case  $A \in H_{2,0}^\infty$ , which means that  $A \in \mathcal{H}(\mathbb{D})$  and  $\lim_{|z| \rightarrow 1^-} (1 - |z|^2)^2 |A(z)| = 0$ . Sequence  $\Lambda \subset \mathbb{D}$  is the zero-sequence of a non-trivial solution of (1) where  $A \in H_{2,0}^\infty$  if and only if  $\Lambda$  is a finite sequence of distinct points in  $\mathbb{D}$ . The fact that all zero-sequences are finite follows from [25, Theorem 1], while the converse assertion is proved by constructing a non-trivial solution of (1), which has finitely many prescribed zeros [12, Section 10].

*Conformally invariant collections of zero-sequences.* Let  $X$  be a space of analytic functions, and let  $\mathcal{Z}(X)$  be the collection of sequences  $\Lambda \subset \mathbb{D}$  for which there exists  $A = A(\Lambda) \in X$  such that (1) admits a non-trivial solution whose zero-sequence is (precisely)  $\Lambda$ . Some parts of the following result are known in another form, see the proof for references.

**Proposition 5.** *The following statements hold:*

- (a)  $\mathcal{Z}(H_2^\infty)$  is conformally invariant.
- (b) If  $\Lambda$  is separated and  $D^+(\Lambda) < 1$ , then  $\Lambda \subset \Lambda'$  for some  $\Lambda' \in \mathcal{Z}(H_2^\infty)$ . Conversely, if  $\Lambda \in \mathcal{Z}(H_2^\infty)$ , then  $\Lambda$  is separated and  $D^+(\Lambda) < \infty$ .
- (c)  $\mathcal{Z}(H_2^\infty)$  contains non-Blaschke sequences. However, if  $\Lambda \in \mathcal{Z}(H_2^\infty)$  and

$$\int_0^{2\pi} \log \text{dist}(e^{i\theta}, \Lambda) d\theta > -\infty, \quad (7)$$

where  $\text{dist}$  is the Euclidean distance, then  $\Lambda$  is a Blaschke sequence.

Let  $K$  be the space which consists of the second derivatives of BMOA functions. Consequently,  $\mathcal{Z}(K)$  is the collection of zero-sequences of non-trivial solutions of (1) induced by those coefficients  $A \in \mathcal{H}(\mathbb{D})$  for which  $d\mu_A(z) = |A(z)|^2(1 - |z|^2)^3 dm(z)$  is a Carleson measure. By Corollary 4,  $\Lambda \in \mathcal{Z}(K)$  if and only if  $\Lambda$  is uniformly separated. The following observations follow from known properties of uniformly separated sequences:

- (a)  $\mathcal{Z}(K)$  is conformally invariant.
- (b) If  $\Lambda \subset \Lambda'$  for some  $\Lambda' \in \mathcal{Z}(K)$ , then  $\Lambda \in \mathcal{Z}(K)$ .
- (c) If  $\Lambda_1, \Lambda_2 \in \mathcal{Z}(K)$ , and  $\Lambda_1 \cup \Lambda_2$  is separated, then  $\Lambda_1 \cup \Lambda_2 \in \mathcal{Z}(K)$ .
- (d) If  $\Lambda = \{z_n\} \in \mathcal{Z}(K)$ , and  $\Lambda' = \{z'_n\} \subset \mathbb{D}$  is a sequence such that  $\sup_n \varrho(z_n, z'_n)$  is sufficiently small, then  $\Lambda' \in \mathcal{Z}(K)$ .
- (e)  $\mathcal{Z}(K)$  contains only Blaschke sequences. However, there are separated Blaschke sequences which are not in  $\mathcal{Z}(K)$ .

By subharmonicity  $K \subset H_2^\infty$ , and hence  $\mathcal{Z}(K) \subset \mathcal{Z}(H_2^\infty)$ . It is curious that  $\mathcal{Z}(H_{2,0}^\infty) \subset \mathcal{Z}(K)$  even though there exist functions  $A \in H_{2,0}^\infty$  for which  $\mu_A$  is not a Carleson measure; typical examples of such functions are given in terms of lacunary series. Note that  $\mathcal{Z}(H_{2,0}^\infty)$  is the collection of finite sequences of distinct points in  $\mathbb{D}$ .

**2.2. Normal functions.** A function  $w$  meromorphic in the unit disc  $\mathbb{D}$  is said to be normal if  $\sup_{z \in \mathbb{D}} (1 - |z|^2) w^\#(z) < \infty$ , where  $w^\# = |w'|/(1 + |w|^2)$  is the spherical derivative of  $w$ . Actually, a meromorphic function  $w$  is normal if and only if  $\{w \circ \varphi : \varphi \text{ conformal automorphism of } \mathbb{D}\}$  is a normal family in  $\mathbb{D}$  (in the sense of Montel). For more details, see [18].

In [16], Lappan gives an answer to a question of Hayman by showing that there exists a non-normal  $w \in \mathcal{H}(\mathbb{D})$  whose Schwarzian derivative

$$S_w = \left( \frac{w''}{w'} \right)' - \frac{1}{2} \left( \frac{w''}{w'} \right)^2$$

belongs to  $H_2^\infty$ . In a subsequent paper [17, Theorem 5] a concrete function having these properties is presented. Recall that, for  $w$  meromorphic in  $\mathbb{D}$ ,  $S_w \in H_2^\infty$  if and only if  $w$  is uniformly locally univalent; see for example

[6, Lemma B] and the references therein. As a consequence of the proof of Theorem 3, we obtain:

**Corollary 6.** *For any  $0 < p \leq 1$ , there exists a non-normal meromorphic function  $w$  in  $\mathbb{D}$  such that  $S_w \in \mathcal{H}(\mathbb{D})$  and  $|S_w(z)|^2(1 - |z|^2)^{2+p} dm(z)$  is a  $p$ -Carleson measure.*

In Corollary 6,  $S_w \in H_2^\infty$  by the subharmonicity of  $|S_w|^2$ . By construction, the function  $w$  has prescribed separated poles  $\Lambda$  such that (5) is a  $p$ -Carleson measure,  $w$  belongs to the Nevanlinna class of meromorphic functions and emerges as a primitive of  $1/f^2$  where  $f \in Q_p \cap H^\infty$  is a solution of (1) for  $A = S_w/2$ .

### 3. PROOFS OF THE RESULTS

The growth space  $H_\alpha^\infty$ , for  $0 \leq \alpha < \infty$ , consists of those  $g \in \mathcal{H}(\mathbb{D})$  for which

$$\|g\|_{H_\alpha^\infty} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |g(z)| < \infty.$$

In particular,  $H^\infty = H_0^\infty$ . Before the proof of Theorem 1, we consider an auxiliary result which resembles the classical Schwarz lemma. See also [3, Lemma 7, p. 209]. The proof of Lemma 7 is presented for convenience of the reader.

**Lemma 7.** *Let  $g \in H_\alpha^\infty$  where  $0 \leq \alpha < \infty$ , and let  $0 < \delta < 1$ . If  $g(z_0) = 0$  for some  $z_0 \in \mathbb{D}$ , then there exists a positive constant  $C = C(\alpha, \delta)$  such that*

$$|g(z)| \leq \frac{C \|g\|_{H_\alpha^\infty} \varrho(z, z_0)}{(1 - |z_0|^2)^\alpha}, \quad z \in \Delta(z_0, \delta). \quad (8)$$

*Proof.* Consider the function  $h(z) = g(z)/((z_0 - z)/(1 - \bar{z}_0 z))$ , and note that  $h \in \mathcal{H}(\mathbb{D})$ . There exists a positive constant  $C^* = C^*(\alpha, \delta)$  such that

$$|h(z)| \leq \frac{|g(z)|}{\varrho(z, z_0)} \leq \frac{\|g\|_{H_\alpha^\infty}}{\delta (1 - |z|^2)^\alpha} \leq \frac{C^* \|g\|_{H_\alpha^\infty}}{\delta (1 - |z_0|^2)^\alpha}, \quad z \in \partial\Delta(z_0, \delta).$$

By the maximum modulus principle this inequality holds for all  $z \in \Delta(z_0, \delta)$ , which implies (8) for  $C = C^*/\delta$ .  $\square$

*Proof of Theorem 1.* Let  $\Lambda$  be a separated sequence for which  $D^+(\Lambda) < 1$ . By [3, Theorem 5, p. 220], there exist a separated sequence  $\Lambda^* \supset \Lambda$  and  $g \in \mathcal{H}(\mathbb{D})$  such that (3) holds for  $\beta = (D^+(\Lambda) + 1)/2 < 1$ . According to [3, Lemma 19, p. 235], we have  $D^-(\Lambda^*) = D^+(\Lambda^*) = \beta$ . Let  $f = ge^h$  where  $h \in \mathcal{H}(\mathbb{D})$  is defined later. The function  $f$  is a solution of (1) with  $A \in \mathcal{H}(\mathbb{D})$ ,

$$A = -\frac{f''}{f} = -\frac{g'' + 2g'h'}{g} - (h')^2 - h'',$$

provided that  $h \in \mathcal{H}(\mathbb{D})$  and  $h'$  satisfies the interpolation property

$$h'(z_n) = -\frac{1}{2} \frac{g''(z_n)}{g'(z_n)}, \quad z_n \in \Lambda^*. \quad (9)$$

In particular, because of (9),  $A$  has a removable singularity at each zero of  $g$ . Since  $g'' \in H_{\beta+2}^\infty$  by Cauchy's integral formula, and

$$\inf_{z_n \in \Lambda^*} (1 - |z_n|^2)^{\beta+1} |g'(z_n)| > 0 \quad (10)$$

by (3), we deduce

$$\sup_{z_n \in \Lambda^*} (1 - |z_n|^2) \left| \frac{g''(z_n)}{g'(z_n)} \right| < \infty. \quad (11)$$

Since  $D^+(\Lambda^*) = \beta < 1$ , [27, Theorem 1.2] and (11) imply that there exists  $h \in \mathcal{H}(\mathbb{D})$  such that  $h' \in H_1^\infty$  and the interpolation property (9) holds.

It remains to prove that  $A \in H_2^\infty$ . Let  $0 < \delta < 1$  be a sufficiently small constant such that the pseudo-hyperbolic discs  $\Delta(z_n, \delta)$  are pairwise disjoint for  $z_n \in \Lambda^*$ . On one hand, by (3) and  $h' \in H_1^\infty$ , we obtain

$$\sup \left\{ (1 - |z|^2)^2 |A(z)| : z \in \mathbb{D} \setminus \bigcup_{z_n \in \Lambda^*} \Delta(z_n, \delta) \right\} < \infty.$$

On the other hand, if  $z \in \Delta(z_n, \delta)$  for some  $z_n \in \Lambda^*$ , then we apply Lemma 7 to  $g'' + 2g'h' \in H_{\beta+2}^\infty$  (which vanishes at all points  $z_n \in \Lambda^*$  by the interpolation property) to deduce that  $(1 - |z|^2)^2 |A(z)|$  is uniformly bounded also for any  $z \in \bigcup_{z_n \in \Lambda^*} \Delta(z_n, \delta)$ . This completes the proof of Theorem 1.  $\square$

The proof of Theorem 1 produces a *non-normal* solution of (1) under the restriction  $A \in H_2^\infty$ . See [6, Theorem 3] for another example. To show that  $f = ge^h$  in the proof of Theorem 1 is non-normal, we argue as follows. For any  $\xi \in \partial\mathbb{D}$ , there exists a subsequence  $\Lambda' = \Lambda'(\xi)$  of  $\Lambda^*$  which converges non-tangentially to  $\xi$ . This follows from [3, Corollary, p. 188] as  $D^-(\Lambda^*) > 0$ . The Makarov law of the iterated logarithm [22, Theorem 8.10] gives

$$\limsup_{r \rightarrow 1^-} \frac{|h(r\xi)|}{\Psi(r)} \leq \|h'\|_{H_1^\infty}, \quad \Psi(r) = \sqrt{\log \frac{1}{1-r} \log \log \log \frac{1}{1-r}}, \quad (12)$$

for almost every  $\xi \in \partial\mathbb{D}$ . Fix  $\xi \in \partial\mathbb{D}$  such that (12) holds, and let  $\Lambda' = \Lambda'(\xi)$  be the corresponding subsequence of  $\Lambda^*$ . By [3, Proposition 1, p. 43],

$$|h(z_n)| \leq |h(|z_n|\xi)| + \frac{\|h'\|_{H_1^\infty}}{2} \log \frac{1 + \varrho(z_n, |z_n|\xi)}{1 - \varrho(z_n, |z_n|\xi)}, \quad z_n \in \Lambda', \quad (13)$$

where  $\sup\{\varrho(z_n, |z_n|\xi) : z_n \in \Lambda'\} < 1$ . If  $z_n \in \Lambda'$  and  $|z_n|$  is sufficiently close to one, then

$$\begin{aligned} (1 - |z_n|^2) |f'(z_n)| &= (1 - |z_n|^2) |g'(z_n)| \exp(\operatorname{Re} h(z_n)) \\ &\gtrsim \frac{1}{(1 - |z_n|^2)^\beta} \exp\left(-(\|h'\|_{H_1^\infty} + 1) \Psi(|z_n|)\right) \end{aligned}$$

by (10), (12) and (13). It follows that  $\sup_{z_n \in \Lambda^*} (1 - |z_n|^2) |f'(z_n)| = \infty$ , and hence  $f$  is non-normal by [10, Proposition 7].

*Proof of Theorem 3.* Let  $0 < p \leq 1$ , and let  $\Lambda \subset \mathbb{D}$  be any separated sequence such that (5) is a  $p$ -Carleson measure. If  $B$  is a Blaschke product

whose zero-sequence is  $\Lambda$ , then  $B \in Q_p \cap H^\infty$  by [4, Theorem 2.2]; the case  $p = 1$  is of course trivial, since  $Q_1 \cap H^\infty = H^\infty$ . Now

$$\inf_{z_n \in \Lambda} (1 - |z_n|^2) |B'(z_n)| = \inf_{z_n \in \Lambda} \prod_{z_k \neq z_n} \left| \frac{z_k - z_n}{1 - \bar{z}_k z_n} \right| > 0$$

by the uniform separation of  $\Lambda$ , and hence

$$\sup_{z_n \in \Lambda} \frac{|B''(z_n)|}{|B'(z_n)|^2} < \infty.$$

Define  $f = Be^h$ , where  $h = Bk$  and  $k \in \mathcal{H}(\mathbb{D})$  is a solution to the interpolation problem

$$k(z_n) = -\frac{B''(z_n)}{2(B'(z_n))^2}, \quad z_n \in \Lambda.$$

By [20, Theorem 1.3], where the condition (b) follows from the fact that (5) is a  $p$ -Carleson measure, or [2, Theorem 3] if  $p = 1$ , we may assume that  $k \in Q_p \cap H^\infty$ . Then, as in the proof of Theorem 1, it follows that  $f$  is a non-trivial solution of (1) where  $A \in \mathcal{H}(\mathbb{D})$  and

$$A = -\frac{f''}{f} = -\frac{B'' + 2B'h'}{B} - (h')^2 - h''. \quad (14)$$

Since  $Q_p \cap H^\infty$  is an algebra, we have  $h \in Q_p \cap H^\infty$  and hence  $f \in Q_p \cap H^\infty$ .

It remains to prove that (6) is a  $p$ -Carleson measure. Let  $0 < \delta < 1$  be a sufficiently small constant such that the pseudo-hyperbolic discs  $\Delta(z_n, \delta)$  are pairwise disjoint for  $z_n \in \Lambda$ . We proceed to verify (4) for  $\mu = \mu_{A,p}$  in two parts. Denote  $\Omega = \mathbb{D} \setminus \bigcup_{z_n \in \Lambda} \Delta(z_n, \delta)$ . Since  $|B|$  is uniformly bounded away from zero on  $\Omega$  and  $h \in H^\infty$ , we obtain

$$|A(z)| = \frac{|f''(z)|}{|f(z)|} \lesssim |f''(z)| |e^{-h(z)}| \lesssim |f''(z)|, \quad z \in \Omega.$$

Consequently,

$$\begin{aligned} & \sup_{a \in \mathbb{D}} \int_{\Omega} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^p d\mu_{A,p}(z) \\ & \lesssim \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f''(z)|^2 (1 - |z|^2)^{2+p} \left( \frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^p dm(z) < \infty \end{aligned}$$

as  $f \in Q_p$ ; see [23, Theorem 3.2]. Only the term  $-(B'' + 2B'h')/B$  in (14) brings us additional trouble on  $\mathbb{D} \setminus \Omega$ , and hence it suffices to show that

$$I = \sup_{a \in \mathbb{D}} \sum_{z_n \in \Lambda} \int_{\Delta(z_n, \delta)} \left| \frac{B''(z) + 2B'(z)h'(z)}{B(z)} \right|^2 \frac{(1 - |z|^2)^{2+p}(1 - |a|^2)^p}{|1 - \bar{a}z|^{2p}} dm(z)$$

is finite. Since  $|B(z)| \gtrsim \varrho(z, z_n)$  for  $z \in \Delta(z_n, \delta)$ , and  $|1 - \bar{a}z| \simeq |1 - \bar{a}z_n|$  for  $z \in \Delta(z_n, \delta)$  and  $a \in \mathbb{D}$  (with comparison constants independent of  $a$ ),

$$I \lesssim \sup_{a \in \mathbb{D}} \sum_{z_n \in \Lambda} \frac{(1 - |z_n|^2)^p (1 - |a|^2)^p}{|1 - \bar{a}z_n|^{2p}} < \infty$$

by Lemma 7 and the fact that (5) is a  $p$ -Carleson measure. This completes the proof of Theorem 3.  $\square$



We may apply the Corona theorem for the algebra  $Q_p \cap H^\infty$  to sharpen a property in [8] (corresponding to the case  $p = 1$ ). Let  $0 < p < 1$ . Assume that  $f_1, f_2 \in Q_p \cap H^\infty$  are linearly independent solutions of (1) such that

$$\inf_{z \in \mathbb{D}} (|f_1(z)| + |f_2(z)|) > 0. \quad (15)$$

By [20, Theorem 1.1] there exist  $g_1, g_2 \in Q_p \cap H^\infty$  such that  $f_1 g_1 + f_2 g_2 \equiv 1$ . Differentiate twice and apply (1) to conclude  $A = f_1 g_1'' + f_2 g_2'' + 2(f_1' g_1' + f_2' g_2')$ . Hence  $d\mu_{A,p}(z) = |A(z)|^2 (1 - |z|^2)^{2+p} dm(z)$  is a  $p$ -Carleson measure.

To construct an example where (15) holds, we consider a method from [12, p. 58]. Let  $0 < p < 1$ , and choose  $g \in Q_p \cap H^\infty$ . Define  $h \in \mathcal{H}(\mathbb{D})$  such that  $h' = e^{-2g}$ , which implies that  $h \in Q_p \cap H^\infty$  and  $h'' + 2g'h' = 0$ . Then, functions  $e^{g+h}, e^{g-h} \in Q_p \cap H^\infty$  are zero-free linearly independent solutions of (1) where  $A = -g'' - (g')^2 - (h')^2 \in \mathcal{H}(\mathbb{D})$ . Estimate (15) follows from the fact that both solutions are uniformly bounded away from zero. In this case it is easy to verify that  $d\mu_{A,p}$  is a  $p$ -Carleson measure.

*Proof of Proposition 5.* (a) Let  $\Lambda \in \mathcal{Z}(H_2^\infty)$  and let  $\tau$  be a conformal automorphism of  $\mathbb{D}$ . We need to prove that  $\tau(\Lambda) \in \mathcal{Z}(H_2^\infty)$ . By assumption, there exists  $A = A(\Lambda) \in H_2^\infty$  such that (1) admits a non-trivial solution  $f$  whose zero-sequence is  $\Lambda$ . Let  $\varphi$  be the inverse of  $\tau$ . Consequently,  $g = (f \circ \varphi)(\varphi')^{-1/2}$  is a non-trivial solution of

$$g'' + Bg = 0, \quad B = (A \circ \varphi)(\varphi')^2,$$

see [21, Lemma 1]. Since  $g$  vanishes precisely on  $\tau(\Lambda)$ , and  $\|B\|_{H_2^\infty} = \|A\|_{H_2^\infty}$  by standard estimates, we have  $\tau(\Lambda) \in \mathcal{Z}(H_2^\infty)$ .

Note that (b) and the first part of (c) follow from Theorem 1, [25, Theorem 3] and Corollary 2. Hence, it suffices to prove the second part of (c). Suppose that  $\Lambda \in \mathcal{Z}(H_2^\infty)$  and (7) holds. Consequently, there exists a coefficient  $A = A(\Lambda) \in H_2^\infty$  such that (1) admits a non-trivial solution  $f$  whose zero-sequence is  $\Lambda$ . By [21, Example 1],  $f \in H_\alpha^\infty$  for any sufficiently large  $\alpha = \alpha(\|A\|_{H_2^\infty}) < \infty$ . According to (7) and [14, Theorem, p. 146],  $\Lambda$  is a Blaschke set of  $H_\alpha^\infty$ , and hence  $\Lambda$  is a Blaschke sequence.  $\square$

*Proof of Corollary 6.* Let  $\Lambda$  be a separated sequence having infinitely many points such that (5) is a  $p$ -Carleson measure, and let  $f$  be the function in the proof of Theorem 3. In particular,  $f = Be^h$  is a non-trivial solution of (1) where  $A \in \mathcal{H}(\mathbb{D})$  and (6) is a  $p$ -Carleson measure. Here  $B \in Q_p \cap H^\infty$  is the Blaschke product corresponding to  $\Lambda$  and  $h \in Q_p \cap H^\infty$ .

Let  $g$  be a solution of (1) which is linearly independent to  $f$ . We may assume that the Wronskian determinant satisfies  $W(f, g) = fg' - f'g \equiv 1$ . If we define  $w = g/f$ , then  $w$  is a locally univalent meromorphic function in  $\mathbb{D}$  such that  $S_w = 2A$  and  $w' = 1/f^2$ . Consequently,  $S_w \in \mathcal{H}(\mathbb{D})$  and  $|S_w(z)|^2 (1 - |z|^2)^{2+p} dm(z)$  is a  $p$ -Carleson measure. It remains to show that  $w$  is non-normal. Since  $\Lambda$  is uniformly separated, there exists a constant  $0 < \delta < 1$  such that  $\delta < (1 - |z_n|^2)|B'(z_n)| \leq 1$  for all  $z_n \in \Lambda$ , and hence

$$|g(z_n)| = \frac{1}{|f'(z_n)|} = \frac{|e^{-h(z_n)}|}{|B'(z_n)|} \simeq 1 - |z_n|^2, \quad z_n \in \Lambda.$$

We conclude that

$$(1 - |z_n|^2) w^\#(z_n) = \frac{1 - |z_n|^2}{|f(z_n)|^2 + |g(z_n)|^2} \simeq \frac{1}{1 - |z_n|^2}, \quad z_n \in \Lambda,$$

which means that  $w$  is non-normal. Finally, we point out that  $w$  belongs to the Nevanlinna class of meromorphic functions by [9, Corollary 3] and [12, Lemma 3].  $\square$

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DEPARTMENT OF PHYSICS AND MATHEMATICS, UNIVERSITY OF EASTERN FINLAND,  
P.O. BOX 111, FI-80101 JOENSUU, FINLAND  
E-mail address: `janne.grohn@uef.fi`